

A NEW METHOD OF FINDING ALL ROOTS OF SIMPLE QUATERNIONIC POLYNOMIALS

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ABSTRACT. In this paper, we provide a new method to find all zeros of polynomials with quaternionic coefficients located on only one side of the powers of the variable (these polynomials are called simple polynomials). This method is much more efficient and much simpler than the known one in [9]. We recover several known results, and deduce several interesting consequences concerning solving equations with all real coefficients or complex coefficients which do not seem to be deduced easily from the results in [9]. We also give a necessary and sufficient condition for a simple quaternionic polynomials to have finitely many solutions (only isolated solutions).

Keywords: quaternion, simple quaternionic polynomial, root

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1. INTRODUCTION

The quaternion algebra plays an important role in many subjects, such as, quaternionic quantum mechanics [1], and signal precessing [2, 8]. Because of the noncommutativity of quaternionic multiplication, solving a quaternionic equation of degree n becomes a challenging problem [3-19].

Niven in [14, 15] made first steps in generalizing the fundamental theorem of algebra onto quaternionic situation which led to the article by Eilenberg and Niven [3] where the existence of roots for a quaternion equation of degree n was proved using strongly topological methods. After that, Topuridze in [20], also with help of topological method, showed that the zero set of polynomials with quaternionic coefficients located on only one side of the power of the single variable (these polynomials are called *simple polynomials*) consists of a finite number of points and Euclidean spheres of corresponding dimension.

Concerning about the computation of roots of a quaternionic polynomial, the first numerically working algorithm to find a root was presented by Serôdio, Pereira, and Vitória [18], and further contributions were made by Serôdio and Siu [19], Pumplün and Walcher [17], De Leo, Ducati, and Leonardi [13], Gentili and Struppa [6], Gentili, Struppa, and Vlacci [7], Gentili and Stoppato [5]. A large bibliography on quaternions in general was given by Gsponer and Hurni in 2006 [4]. Recently, Janovská and Opfer presented a method in [9] for producing all zeros of a simple quaternionic

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polynomial, by using the real companion polynomial introduced for the first time by Niven [14], the number one introduced in [16], and the presentation of the powers of a quaternion as a real, linear combination of the quaternion. Let us recall some notions first.

Throughout this paper, let \mathbb{N} be the set of positive integers, \mathbb{R} the real number field, $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\mathbf{i}$ the complex number field, and \mathbb{H} the skew-field of real quaternions, that is, any element of \mathbb{H} is of the form $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_0 + a_1\mathbf{i}) + (a_2 + a_3\mathbf{i})\mathbf{j}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are usual quaternionic imaginary units, and $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and the \mathbb{R} -bilinear product is determined by $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. For the above quaternion q , we denote by $\operatorname{Re} q$ the real part of q , by $|q|$ the module of q (i.e. $|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$), and by $[q]$ the *conjugate class* of q (i.e. $[q] = \{aqa^{-1} | a \in \mathbb{H}, a \neq 0\}$). We confirm that a quaternionic polynomial with the coefficients on the same side of the power of the single variable are called a *simple quaternionic polynomial* or a *simple polynomial*.

Definition 1.1. Let $z_0 \in \mathbb{H}$ be a zero of a simple polynomial $p(z)$ as given on the left-hand side in (2.1) (i.e., $p(z_0) = 0$). If z_0 is not real and has the property that $p(z) = 0$ for all $z \in [z_0]$, then we will say that z_0 is a *spherical zero* of $p(z)$. If z_0 is real or is not a spherical zero, it is called an *isolated zero* of $p(z)$.

In the present paper, we provide a new method for finding all zeros of simple polynomials $p(z)$ of arbitrary degree n (Theorems 1 and 4). Our proof is based on two well-known techniques: the presentation of a quaternion as a 2×2 complex matrix, and the Jordan canonical form of a complex matrix. We first write $p(z)$ so that its constant term is 1 or 0. Then introduce *derived polynomials* $f_1(t)$ and $f_2(t)$ of $p(z)$ which have complex coefficients, where t is a real variable, such that $p(t) = f_1(t) + f_2(t)\mathbf{j}$; and define the *discriminant polynomial* $\tilde{p}(t) = f_1(t)\bar{f}_1(t) + f_2(t)\bar{f}_2(t)$ of $p(z)$ which is a polynomial with real coefficients, where t is considered as a complex variable, the polynomials $\bar{f}_1(t)$ and $\bar{f}_2(t)$ are obtained by only taking the conjugate coefficients of $f_1(t)$ and $f_2(t)$ respectively. Then all zeros of $p(z)$ can be obtained from complex zeros of the discriminant polynomial $\tilde{p}(t)$. More precisely, let $z_0 \in \mathbb{C}$ such that $\tilde{p}(z_0) = 0$. If z_0 is real then it is an isolated zero of $p(z)$. If z_0 is not real and $f_1(z_0) = f_2(z_0) = \bar{f}_1(z_0) = \bar{f}_2(z_0) = 0$ then it is a spherical zero of $p(z)$. If z_0 is not real and at least one of $f_1(z_0), f_2(z_0), \bar{f}_1(z_0), \bar{f}_2(z_0)$ is not zero, let $\begin{pmatrix} a \\ b \end{pmatrix}$ be a unit complex solution of the linear system

$$\begin{pmatrix} f_2(\bar{z}_0) & f_1(\bar{z}_0) \\ \bar{f}_1(\bar{z}_0) & -\bar{f}_2(\bar{z}_0) \end{pmatrix} X = 0.$$

Then corresponding to this z_0 we have an isolated solution for $p(t)$:

$$|a|^2 z_0 + |b|^2 \bar{z}_0 - 2b\bar{a}(\operatorname{Im} z_0)\mathbf{k}.$$

From the above three cases we obtain all zeros of $p(z)$.

The paper is organized as follows. In Section 2, we prove that our methods for solving a simple polynomial equation in Theorems 1 and 4 are valid. Then we recover several known results, and deduce several very interesting

consequences concerning solving equations with all real coefficients or complex coefficients which do not seem to be deduced easily from the results in [9] (see Corollary 2 and Corollary 3). We also give a necessary and sufficient condition for a simple quaternionic polynomials to have finitely many solutions (only isolated solutions). In Section 3 we give an algorithm to find all zeros of a simple quaternionic equation, based upon our Theorems 1 and 4. In Section 4 we give three examples. In particular, we use our method to redo Example 3.8 in [9]. In Section 5, we make some necessary numerical considerations and compare our algorithm with that in [9].

2. FINDING ALL ZEROS OF A SIMPLE POLYNOMIAL

We consider the simple quaternionic polynomial equation:

$$\text{eq:simple} \quad (2.1) \quad q_n x^n + \cdots + q_1 x + q_0 = 0 \quad (q_n \neq 0),$$

where $x \in \mathbb{H}$ is the variable, $n \in \mathbb{N}$ and q_i ($i = 0, \dots, n$) $\in \mathbb{H}$ are given. If $q_0 \neq 0$, then Eq.(2.1) can be written as

$$q_0^{-1} q_n x^n + \cdots + q_0^{-1} q_1 x + 1 = 0.$$

Hence, in order to solve Eq.(2.1), it suffices to solve the following equation

$$\text{eq:simple 1} \quad (2.2) \quad p_n x^n + \cdots + p_1 x + d_0 = 0,$$

where p_i ($i = 1, \dots, n$) $\in \mathbb{H}$, $p_n \neq 0$, $d_0 = 0$ or 1 . We simply denote the left-hand side of (2.2) as $p(x)$.

Let $\sigma : \mathbb{H} \rightarrow \mathbb{C}^{2 \times 2}$, $q = z_1 + z_2 \mathbf{j} \mapsto \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$, where $z_1, z_2 \in \mathbb{C}$.

Then σ is an \mathbb{R} -algebra monomorphism from \mathbb{H} to $\mathbb{C}^{2 \times 2}$. Sometimes, this monomorphism is also named as the derived mapping of \mathbb{H} , and $\sigma(q)$ is denoted by q^σ . Obviously, $a^\sigma = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for any $a \in \mathbb{R}$.

Let $p_i^\sigma = \begin{pmatrix} t_1^{(i)} & t_2^{(i)} \\ -t_2^{(i)} & t_1^{(i)} \end{pmatrix}$ for $i = 1, \dots, n$, where $t_1^{(i)}, t_2^{(i)} \in \mathbb{C}$. Then (2.2) becomes the following matrix equation in the matrix variable $Y = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$ ($z_1, z_2 \in \mathbb{C}$):

$$\text{eq:matrix} \quad (2.3) \quad \begin{pmatrix} t_1^{(n)} & t_2^{(n)} \\ -t_2^{(n)} & t_1^{(n)} \end{pmatrix} Y^n + \cdots + \begin{pmatrix} t_1^{(1)} & t_2^{(1)} \\ -t_2^{(1)} & t_1^{(1)} \end{pmatrix} Y + \begin{pmatrix} d_0 & 0 \\ 0 & d_0 \end{pmatrix} = 0.$$

Now we introduce a matrix polynomial $P(t)$ in real variable t (considered as a real number) as follows:

$$\text{P(t)} \quad (2.4) \quad P(t) \equiv p_n^\sigma t^n + \cdots + p_1^\sigma t + d_0 I,$$

where I is the 2×2 identity matrix.

Write $P(t) = Q(t)(tI - Y) + P_l(Y)$, where

$$P_l(Y) = p_n^\sigma Y^n + \cdots + p_1^\sigma Y + d_0 I,$$

$$Q(t) = p_n^\sigma t^{n-1} + (p_{n-1}^\sigma + p_n^\sigma Y) t^{n-2} + \cdots + (p_1^\sigma + p_2^\sigma Y + \cdots + p_n^\sigma Y^{n-1}).$$

If $P_l(Y) = 0$, then $P(t) = Q(t)(tI - Y)$, and hence

$$\det P(t) = \det Q(t) \det(tI - Y) = \chi_Y(t) \det Q(t),$$

where $\chi_Y(t)$ is the characteristic polynomial of Y .

Set $\tilde{p}(t) \equiv \det P(t)$. By Cayley-Hamilton Theorem, we see that $\tilde{p}(Y) = 0$ for every Y satisfying $P_l(Y) = 0$.

Notice that, $\tilde{p}(t) \equiv \det P(t)$ is a polynomial in real variable t of degree $2n$ with real coefficients, since $\tilde{p}(t) = \det(p_n t^n + \cdots + p_1 t + d_0)^\sigma$, and $\det(p_n t^n + \cdots + p_1 t + d_0)^\sigma \geq 0$ for any real value of t . Then

$$\boxed{\mathbf{p}(\mathbf{t})} \quad (2.5) \quad \tilde{p}(t) = b(t - \xi_1)^{2r_1} \cdots (t - \xi_s)^{2r_s}.$$

$$\cdot (t - \eta_1)^{s_1} (t - \bar{\eta}_1)^{s_1} \cdots (t - \eta_k)^{s_k} (t - \bar{\eta}_k)^{s_k},$$

where b is a real number, ξ_1, \dots, ξ_s are distinct real numbers, $\eta_1, \bar{\eta}_1, \dots, \eta_k, \bar{\eta}_k$ are distinct nonreal complex numbers, and $r_1, r_2, \dots, r_s, s_1, \dots, s_k \in \mathbb{N}$. It is clear that $s + k \leq n$.

Now suppose Y is a solution of Equation (2.3). Then, $\tilde{p}(Y) = 0$. Since Y is of form $\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$ where $z_1, z_2 \in \mathbb{C}$, then $\chi_Y(t) = (t - z_1)(t - \bar{z}_1) + z_2 \bar{z}_2 \geq 0$ for all real values of t . Consequently, Y has two equal real eigenvalues or two conjugate complex eigenvalues. Hence, its Jordan canonical form, J_Y , has to be of form aI or $\begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}$ where $a \in \mathbb{R}$ and c is a complex with nonzero imaginary part. Using (2.5) we know that, for a solution Y of Equation (2.3), J_Y has to be one of the following $s + k$ matrices (up to the order of the diagonal entries):

$$\boxed{\mathbf{J_Y}} \quad (2.6) \quad \xi_1 I, \dots, \xi_s I, \begin{pmatrix} \eta_1 & 0 \\ 0 & \bar{\eta}_1 \end{pmatrix}, \dots, \begin{pmatrix} \eta_k & 0 \\ 0 & \bar{\eta}_k \end{pmatrix}.$$

Next we will prove that each of the above cases can occur.

Case 1: $J_Y = \xi_i I$ for $i = 1, \dots, s$.

In this case $Y = J_Y$. Clearly, Y is a solution of Equation (2.3) iff $p_n^\sigma \xi_i^n + \cdots + p_1^\sigma \xi_i + d_0 I = 0$, iff ξ_i is a common real root of both

$$t_1^{(n)} x^n + \cdots + t_1^{(1)} x + d_0 = 0$$

and

$$t_2^{(n)} x^n + \cdots + t_2^{(1)} x = 0.$$

But ξ_i is a real root of $\tilde{p}(t)$, the equalities $t_1^{(n)} \xi_i^n + \cdots + t_1^{(1)} \xi_i + d_0 = 0$ and $t_2^{(n)} \xi_i^n + \cdots + t_2^{(1)} \xi_i = 0$ hold naturally. Hence, $\xi_i I$ ($i = 1, \dots, s$) is a solution of the equation (2.3).

Case 2: $J_Y = \begin{pmatrix} \eta_i & 0 \\ 0 & \bar{\eta}_i \end{pmatrix}$ for $i = 1, \dots, k$.

We may assume that (see [22])

$$Y = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} \eta_i & 0 \\ 0 & \bar{\eta}_i \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}^{-1}$$

for some $z_1, z_2 \in \mathbb{C}$ with $|z_1|^2 + |z_2|^2 \neq 0$. Then Y is a solution of (2.3) if and only if

$$p_n^\sigma \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} \eta_i^n & 0 \\ 0 & \bar{\eta}_i^n \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}^{-1} + \cdots \\ + p_1^\sigma \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} \eta_i & 0 \\ 0 & \bar{\eta}_i \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}^{-1} + d_0 I = 0,$$

i.e.,

$$\boxed{\text{eq:2.4}} \quad (2.7) \quad p_n^\sigma \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} \eta_i^n & 0 \\ 0 & \bar{\eta}_i^n \end{pmatrix} + \cdots \\ + p_1^\sigma \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} \eta_i & 0 \\ 0 & \bar{\eta}_i \end{pmatrix} + d_0 \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = 0.$$

In other words, for any nonzero solution $\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$ of (2.7), we obtain a solution for (2.3) of the following form:

$$\boxed{\text{eq:2.5}} \quad (2.8) \quad \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} \eta_i & 0 \\ 0 & \bar{\eta}_i \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}^{-1}.$$

We write (2.7) into two equations

$$\begin{cases} t_1^{(n)} \eta_i^n z_1 + t_2^{(n)} \eta_i^n (-\bar{z}_2) + \cdots + t_1^{(1)} \eta_i z_1 + t_2^{(1)} \eta_i (-\bar{z}_2) + d_0 z_1 = 0 \\ t_1^{(n)} \bar{\eta}_i^n z_2 + t_2^{(n)} \bar{\eta}_i^n \bar{z}_1 + \cdots + t_1^{(1)} \bar{\eta}_i z_2 + t_2^{(1)} \bar{\eta}_i \bar{z}_1 + d_0 z_2 = 0, \end{cases}$$

i.e.,

$$\boxed{\text{eq:2.6}} \quad (2.9) \quad \begin{cases} (t_2^{(n)} \bar{\eta}_i^n + \cdots + t_2^{(1)} \bar{\eta}_i) \bar{z}_1 + (t_1^{(n)} \bar{\eta}_i^n + \cdots + t_1^{(1)} \bar{\eta}_i + d_0) z_2 = 0 \\ (t_1^{(n)} \bar{\eta}_i^n + \cdots + t_1^{(1)} \bar{\eta}_i + d_0) \bar{z}_1 - (t_2^{(n)} \bar{\eta}_i^n + \cdots + t_2^{(1)} \bar{\eta}_i) z_2 = 0. \end{cases}$$

Considering (2.9) as a linear system in the variables $X = \begin{pmatrix} \bar{z}_1 \\ z_2 \end{pmatrix}$, we see that its determinant of the coefficient matrix is

$$D = \begin{vmatrix} t_2^{(n)} \bar{\eta}_i^n + \cdots + t_2^{(1)} \bar{\eta}_i & t_1^{(n)} \bar{\eta}_i^n + \cdots + t_1^{(1)} \bar{\eta}_i + d_0 \\ t_1^{(n)} \bar{\eta}_i^n + \cdots + t_1^{(1)} \bar{\eta}_i + d_0 & -(t_2^{(n)} \bar{\eta}_i^n + \cdots + t_2^{(1)} \bar{\eta}_i) \end{vmatrix}.$$

Since η_i and $\bar{\eta}_i$ are roots of

$$\tilde{p}(t) = \det P(t) = \begin{vmatrix} t_1^{(n)} t^n + \cdots + t_1^{(1)} t + d_0 & t_2^{(n)} t^n + \cdots + t_2^{(1)} t \\ -(t_2^{(n)} t^n + \cdots + t_2^{(1)} t) & t_1^{(n)} t^n + \cdots + t_1^{(1)} t + d_0 \end{vmatrix},$$

it follows that $D = 0$, which shows that (2.9) always has nonzero solutions. Since all coefficients of (2.9) are known, the solution set of (2.9) in $X = \begin{pmatrix} \bar{z}_1 \\ z_2 \end{pmatrix}$ can be given clearly, which will be denoted by Γ_i .

If η_i and $\bar{\eta}_i$ simultaneously satisfy $t_1^{(n)} x^n + \cdots + t_1^{(1)} x + d_0 = 0$ and $t_2^{(n)} x^n + \cdots + t_2^{(1)} x = 0$, then (2.9) becomes trivial, and $\Gamma_i = \mathbb{C}^{2 \times 1}$. Consequently, any element of the form (2.8) is a solution of (2.3).

Now suppose (2.9) is nontrivial. Then Γ_i is of dimension 1, and $\Gamma_i = \left\{ z \begin{pmatrix} a^{(i)} \\ b^{(i)} \end{pmatrix} \mid z \in \mathbb{C} \right\}$, in which $\begin{pmatrix} a^{(i)} \\ b^{(i)} \end{pmatrix}$ is a fixed nonzero solution of (2.9)

with $|a^{(i)}|^2 + |b^{(i)}|^2 = 1$. Up to now, we have actually provided a method to find all the roots of (2.1) in \mathbb{H} . To summarize our result as a theorem, we need to introduce some notions.

Let $p_i = t_1^{(i)} + t_2^{(i)} \mathbf{j} \in \mathbb{H}$ for $i = 1, \dots, n$ where $t_1^{(i)}, t_2^{(i)} \in \mathbb{C}$. We call the following four polynomials *the derived polynomials* of Equation (2.2):

$$\begin{aligned} f_1(t) &= t_1^{(n)} t^n + \dots + t_1^{(1)} t + d_0, & f_2(t) &= t_2^{(n)} t^n + \dots + t_2^{(1)} t; \\ \bar{f}_1(t) &= \overline{t_1^{(n)}} t^n + \dots + \overline{t_1^{(1)}} t + d_0, & \bar{f}_2(t) &= \overline{t_2^{(n)}} t^n + \dots + \overline{t_2^{(1)}} t. \end{aligned}$$

We define the *discriminant polynomial* of Equation (2.2) as $\tilde{p}(t) = f_1(t)\bar{f}_1(t) + f_2(t)\bar{f}_2(t)$. We factor it as in (2.5). Remark that these two $\tilde{p}(t)$ are essentially equal. Introduce sets T_1 and T_2 as follows

$$T_1 = \{\eta \in \{\eta_1, \dots, \eta_k\} \mid f_1(\eta) = f_2(\eta) = \bar{f}_1(\eta) = \bar{f}_2(\eta) = 0\},$$

$$T_2 = \{\eta_1, \dots, \eta_k\} \setminus T_1.$$

Now we can state our main result

theorem1

Theorem 1. *With the above notations, the solution set of (2.2) over \mathbb{H} is*

formula1

$$(2.10) \quad \{\xi_1, \dots, \xi_s\} \dot{\cup}_{\eta_i \in T_2} \{\omega_i\} \dot{\cup}_{\eta_i \in T_1} [\eta_i],$$

where ω_i takes

$$(2.11) \quad \frac{1}{|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2} \{ |f_2(\eta_i)|^2 \eta_i + |f_1(\eta_i)|^2 \bar{\eta}_i - 2f_2(\eta_i) \overline{f_1(\eta_i)} (\text{Im} \eta_i) \mathbf{k} \}$$

as its value if $|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2 \neq 0$, otherwise takes

$$(2.12) \quad \frac{1}{|f_1(\bar{\eta}_i)|^2 + |f_2(\bar{\eta}_i)|^2} \{ |f_1(\bar{\eta}_i)|^2 \eta_i + |f_2(\bar{\eta}_i)|^2 \bar{\eta}_i + 2f_2(\bar{\eta}_i) \overline{f_1(\bar{\eta}_i)} (\text{Im} \eta_i) \mathbf{k} \}$$

as its value, in which $\text{Im} \eta_i$ means the imaginary part (real number) of η_i . Moreover, the union of the first two parts in the above set is the set of isolated solutions and the third part of the above set is the set of spherical solutions.

Proof. To finish the proof we need to continue the argument on the case when (2.9) is nontrivial, i.e., $\eta_i \in T_2$. For any $\eta_i \in T_2$, let $\begin{pmatrix} a^{(i)} \\ b^{(i)} \end{pmatrix}$ be

a nonzero complex solution of the system $\begin{pmatrix} f_2(\bar{\eta}_i) & f_1(\bar{\eta}_i) \\ \bar{f}_1(\bar{\eta}_i) & -\bar{f}_2(\bar{\eta}_i) \end{pmatrix} X = 0$ with $|a^{(i)}|^2 + |b^{(i)}|^2 = 1$. Since the set of nonzero solutions of (2.9) is unknown $X = \begin{pmatrix} \bar{z}_1 \\ z_2 \end{pmatrix}$ is $\left\{ l \begin{pmatrix} a^{(i)} \\ b^{(i)} \end{pmatrix} \mid l \neq 0, l \in \mathbb{C} \right\}$, the solutions of (2.3) corresponding to

the Jordan canonical form $\begin{pmatrix} \eta_i & 0 \\ 0 & \bar{\eta}_i \end{pmatrix}$ are

$$\begin{aligned} & \begin{pmatrix} \overline{la^{(i)}} & lb^{(i)} \\ -\overline{lb^{(i)}} & la^{(i)} \end{pmatrix} \begin{pmatrix} \eta_i & 0 \\ 0 & \bar{\eta}_i \end{pmatrix} \begin{pmatrix} \overline{la^{(i)}} & lb^{(i)} \\ -\overline{lb^{(i)}} & la^{(i)} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \overline{a^{(i)}} & b^{(i)} \\ -\overline{b^{(i)}} & a^{(i)} \end{pmatrix} \begin{pmatrix} \eta_i & 0 \\ 0 & \bar{\eta}_i \end{pmatrix} \begin{pmatrix} \overline{a^{(i)}} & b^{(i)} \\ -\overline{b^{(i)}} & a^{(i)} \end{pmatrix}^{-1}, \end{aligned}$$

which is actually one value, and in \mathbb{H} which can be written as

$$\begin{aligned} (\overline{a^{(i)}} + b^{(i)}\mathbf{j})\eta_i(\overline{a^{(i)}} + b^{(i)}\mathbf{j})^{-1} &= (\overline{a^{(i)}} + b^{(i)}\mathbf{j})\eta_i(a^{(i)} - b^{(i)}\mathbf{j}) \\ &= |a^{(i)}|^2\eta_i + |b^{(i)}|^2\overline{\eta_i} + \overline{a^{(i)}}b^{(i)}(\overline{\eta_i} - \eta_i)\mathbf{j}, \end{aligned}$$

which can not be a real number. Therefore, the solution set of $p_n x^n + \dots + p_1 x + d_0 = 0$ in the skew-field \mathbb{H} is

formula2b

(2.13)

$$\{\xi_1, \dots, \xi_s\} \dot{\cup}_{\eta_i \in T_2} \left\{ |a^{(i)}|^2\eta_i + |b^{(i)}|^2\overline{\eta_i} - 2\overline{a^{(i)}}b^{(i)}(\text{Im } \eta_i)\mathbf{k} \right\} \dot{\cup}_{\eta_i \in T_1} [\eta_i].$$

Note that $|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2$ and $|f_1(\overline{\eta_i})|^2 + |f_2(\overline{\eta_i})|^2$ can not be 0 simultaneously for $\eta_i \in T_2$, then we can take

$$\begin{pmatrix} a^{(i)} \\ b^{(i)} \end{pmatrix} = \begin{pmatrix} \frac{\overline{f_2(\eta_i)}}{\sqrt{|\overline{f_1(\eta_i)}|^2 + |\overline{f_2(\eta_i)}|^2}} \\ \frac{f_1(\eta_i)}{\sqrt{|\overline{f_1(\eta_i)}|^2 + |\overline{f_2(\eta_i)}|^2}} \end{pmatrix}$$

for the case $|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2 \neq 0$, otherwise we take

$$\begin{pmatrix} a^{(i)} \\ b^{(i)} \end{pmatrix} = \begin{pmatrix} \frac{f_1(\overline{\eta_i})}{\sqrt{|f_1(\overline{\eta_i})|^2 + |f_2(\overline{\eta_i})|^2}} \\ \frac{-f_2(\overline{\eta_i})}{\sqrt{|f_1(\overline{\eta_i})|^2 + |f_2(\overline{\eta_i})|^2}} \end{pmatrix}.$$

After manipulations, the set in (2.13) becomes

$$\{\xi_1, \dots, \xi_s\} \dot{\cup}_{\eta_i \in T_2} \{\omega_i\} \dot{\cup}_{\eta_i \in T_1} [\eta_i],$$

where ω_i takes

$$\frac{1}{|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2} \{ |f_2(\eta_i)|^2\eta_i + |f_1(\eta_i)|^2\overline{\eta_i} - 2f_2(\eta_i)\overline{f_1(\eta_i)}(\text{Im } \eta_i)\mathbf{k} \}$$

as its value if $|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2 \neq 0$; otherwise it takes

$$\frac{1}{|f_1(\overline{\eta_i})|^2 + |f_2(\overline{\eta_i})|^2} \{ |f_1(\overline{\eta_i})|^2\eta_i + |f_2(\overline{\eta_i})|^2\overline{\eta_i} + 2f_2(\overline{\eta_i})\overline{f_1(\overline{\eta_i})}(\text{Im } \eta_i)\mathbf{k} \}$$

as its value.

Finally, it is clear that $[\eta_i]$ contains no real numbers for $\eta_i \in T_1$. This completes the proof. \square

Note that there is no repetition in the solution set given in (2.10) and one can use (2.11) or (2.12) for ω_i if

$$(|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2)(|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2) \neq 0.$$

Theorem 1 shows, once we get a decomposition with the form (2.5) of the discriminant polynomial $\tilde{p}(t)$, then we can produce all roots of the quaternionic equation (2.2) by formula (2.10).

From Theorem 1 and the arguments before Theorem 1, we can easily see the following well-known results.

corollary2

Corollary 2. (a). Any simple quaternionic equation $q_n x^n + \dots + q_1 x + q_0 = 0$ ($q_n \neq 0$) has a root in \mathbb{H} .

(b). The simple quaternionic equation $q_n x^n + \dots + q_1 x + q_0 = 0$ ($q_n \neq 0$) has a finite number of roots in \mathbb{H} iff it has at most n distinct roots in \mathbb{H} .

- (c). The roots of $q_n x^n + \cdots + q_1 x + q_0 = 0 (q_n \neq 0)$ are distributed in at most n conjugate classes, and there are at most n real roots among them.

Proof. (a) is obvious.

(b) From Theorem 1 we need to only show that $[\eta_i]$ is an infinite set if $\eta_i \in T_1$. It is well-known from [21] that $u_1, u_2 \in \mathbb{H}$ are conjugate (i.e., there exists nonzero $q \in \mathbb{H}$ such that $u_1 = qu_2 q^{-1}$) iff $\operatorname{Re} u_1 = \operatorname{Re} u_2$ and $|u_1| = |u_2|$. Since $\eta_i \in T_1$ has a nonzero imaginary part, $[\eta_i]$ is an infinite set. Thus $q_n x^n + \cdots + q_1 x + q_0 = 0$ has a finite number of roots in \mathbb{H} iff $T_1 = \emptyset$, iff $p_n x^n + \cdots + p_1 x + d_0 = 0$ has at most n roots in \mathbb{H} since $s+k \leq n$ (See (2.5) for the notations).

(c) follows from $s+k \leq n$. \square

Now we give a quick method to solve simple quaternionic polynomials with all real coefficients or with all complex coefficients. These results do not seem to be deduced easily from the results in [9].

corollary3

Corollary 3. (a). If all q_i in $q_n x^n + \cdots + q_1 x + q_0 = 0 (q_n \neq 0)$ are real numbers and the solution set of this equation in \mathbb{C} is $\{\xi_1, \dots, \xi_s, \zeta_1, \bar{\zeta}_1, \dots, \zeta_t, \bar{\zeta}_t\}$, where ξ_1, \dots, ξ_s are distinct real numbers, ζ_1, \dots, ζ_t are distinct nonreal complex numbers, then the solution set of this equation in \mathbb{H} is

$$\{\xi_1, \dots, \xi_s, \} \cup [\zeta_1] \cup \cdots \cup [\zeta_t].$$

- (b). More generally, if all q_i in $q_n x^n + \cdots + q_1 x + q_0 = 0 (q_n \neq 0)$ are complex numbers, and the solution set of this equation in \mathbb{C} is $\{\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_k, \zeta_1, \bar{\zeta}_1, \dots, \zeta_t, \bar{\zeta}_t\}$, where ξ_1, \dots, ξ_s are distinct real numbers, $\eta_1, \dots, \eta_k, \zeta_1, \dots, \zeta_t$ are distinct nonreal complex numbers (each $\bar{\eta}_i$ is no longer the root of this equation), then the solution set of this equation in \mathbb{H} is

$$\{\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_k, \} \cup [\zeta_1] \cup \cdots \cup [\zeta_t].$$

- (c). Let $f_1(t)$, $f_2(t)$, $\bar{f}_1(t)$ and $\bar{f}_2(t)$ be the derived polynomials for (2.2). Then (2.2) has finitely many solutions iff the complex polynomial $\gcd(f_1(t), f_2(t), \bar{f}_1(t), \bar{f}_2(t))$ has no nonreal complex root iff the complex polynomial $\gcd(f_1(t), f_2(t))$ has no nonreal conjugate complex roots.

Proof. (a) This is a special case of Part (b).

(b) When the equation considered has only complex coefficients, two of the derived polynomials are $f_1(t) = q_n t^n + \cdots + q_1 t + q_0$ (up to a complex scalar), and $f_2(t) = 0$. So, the roots of the discriminant polynomial are $\xi_1, \dots, \xi_s, \eta_1, \bar{\eta}_1, \dots, \eta_k, \bar{\eta}_k, \zeta_1, \bar{\zeta}_1, \dots, \zeta_t, \bar{\zeta}_t$. For each η_i ($i = 1, \dots, k$), since $\bar{f}_1(\eta_i) \neq 0$, η_i is in T_2 . We can take $\begin{pmatrix} a^{(i)} \\ b^{(i)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as the unit complex solution of the system $\begin{pmatrix} f_2(\bar{\eta}_i) & f_1(\bar{\eta}_i) \\ f_1(\bar{\eta}_i) & -\bar{f}_2(\bar{\eta}_i) \end{pmatrix} X = 0$. Then, $|a^{(i)}|^2 \eta_i + |b^{(i)}|^2 \bar{\eta}_i - 2b^{(i)} \overline{a^{(i)}} (\operatorname{Im} \eta_i) \mathbf{k} = \eta_i$. It is easy to see that $\zeta_i \in T_1$. This completes the proof.

(c) Suppose c is a nonreal complex root of $\gcd(f_1(t), f_2(t), \bar{f}_1(t), \bar{f}_2(t))$. Then both c and \bar{c} are roots of the discriminant polynomial, $f_1(c) = f_2(c) = \bar{f}_1(c) = \bar{f}_2(c) = 0$ and $f_1(\bar{c}) = f_2(\bar{c}) = \bar{f}_1(\bar{c}) = \bar{f}_2(\bar{c}) = 0$, which implies that at least one of c, \bar{c} is in T_1 . Thus, $T_1 = \emptyset$ iff $\gcd(f_1(t), f_2(t), \bar{f}_1(t), \bar{f}_2(t))$ has no nonreal complex root. From Theorem 1 we see that (2.2) has finitely many solutions iff $T_1 = \emptyset$. The conclusions in the corollary follow easily. \square

Now we can give a simplified version of Theorem 1.

Theorem 4. Consider the simple quaternionic equation $p(x) := p_n x^n + \cdots + p_1 x + d_0 = 0$, where $p_i \in \mathbb{H}$ with $p_n \neq 0$ and $d_0 = 0$ or 1. We write $p(t) = g(t)(g_1(t) + g_2(t)\mathbf{j})$ where t is considered as a real variable, $g, g_1, g_2 \in \mathbb{C}[t]$ with $\gcd(g_1, g_2) = 1$. Let the complex solution sets for $g(t)$ and $\bar{g}(t) = g_1(t)\bar{g}_1(t) + g_2(t)\bar{g}_2(t)$ are

first (2.14) $\{\xi_1, \dots, \xi_s; \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \dots, \lambda_t, \bar{\lambda}_t; \eta_1, \dots, \eta_{k_1}\},$

sec (2.15) $\{\eta_{k_1+1}, \bar{\eta}_{k_1+1}, \dots, \eta_k, \bar{\eta}_k\}$

respectively. We may assume that ξ_1, \dots, ξ_s are distinct real numbers; $\eta_1, \dots, \eta_{k_1}, \eta_{k_1+1}, \bar{\eta}_{k_1+1}, \dots, \eta_k, \bar{\eta}_k, \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \dots, \lambda_t, \bar{\lambda}_t$ are distinct nonreal complex numbers after deleting those η_i 's and $\bar{\eta}_i$'s in (2.15) if one of them appears in the set (2.14). Then the quaternionic solution set of $p(x)$ is

formula2 (2.16) $\{\xi_1, \dots, \xi_s; \omega_1, \dots, \omega_k\} \dot{\cup}_{i=1}^t [\lambda_i],$

where

$$\omega_i = \frac{1}{|g_1(\bar{\eta}_i)|^2 + |g_2(\bar{\eta}_i)|^2} \{ |g_2(\bar{\eta}_i)|^2 \bar{\eta}_i + |g_1(\bar{\eta}_i)|^2 \eta_i + 2g_2(\bar{\eta}_i) \overline{g_1(\bar{\eta}_i)} (\text{Im} \eta_i) \mathbf{k} \}.$$

Proof. We continue to use the notations in Theorem 1. We know that $f_1(t) = g(t)g_1(t)$, $f_2(t) = g(t)g_2(t)$, $\tilde{p}(t) = g(t)\bar{g}(t)\tilde{g}(t)$, and $\tilde{g}(t)$ has no real root.

From Theorem 1 we see that $\{\xi_1, \dots, \xi_s\} \dot{\cup}_{i=1}^{t_1} [\lambda_i]$ are zeros of $p(x)$. All other zeros come from $\{\eta_1, \dots, \eta_{k_1}, \eta_{k_1+1}, \bar{\eta}_{k_1+1}, \dots, \eta_k, \bar{\eta}_k\}$. For each η_i , we see that $|f_1(\bar{\eta}_i)|^2 + |f_2(\bar{\eta}_i)|^2 \neq 0$. Then using (2.12) in Theorem 1 and simplifying we obtain ω_i . This completes the proof. \square

Remark that the above theorem simplifies the computation for finding all zeros of $p(x)$, and the following known result (see [16]) can follow easily from the above theorem.

Corollary 5. The spherical zeros of simple quaternionic polynomial $p(x) := p_n x^n + \cdots + p_1 x + p_0$ ($p_i \in \mathbb{H}, p_n \neq 0$) are distributed in at most $\text{INT}(\frac{n}{2})$ conjugate classes, where $\text{INT}(\frac{n}{2})$ means the integral function value at $\frac{n}{2}$.

3. ALGORITHM

Based on our Theorem 1, we now can give an algorithm to solve the quaternionic equation $q_n x^n + \cdots + q_1 x + q_0 = 0$ ($q_n \neq 0$), as follows.

Algorithm 1 (for solving the simple quaternionic polynomial equation $q_n x^n + \cdots + q_1 x + q_0 = 0$)

Step 1. Write the equation as $p_n x^n + \cdots + p_1 x + d_0 = 0$ with $d_0 = 0$ or 1 (in fact, if $q_0 \neq 0$, simply multiply the equation by q_0^{-1} on the left).

Write $p_i = t_1^{(i)} + t_2^{(i)}\mathbf{j}$ for $i = 1, \dots, n$ with $t_1^{(i)}, t_2^{(i)} \in \mathbb{C}$. Find the derived polynomials and discriminant polynomial of $p_n x^n + \dots + p_1 x + d_0 = 0$:

$$\begin{aligned} f_1(t) &= \frac{t_1^{(n)}}{t_1^{(1)}} t^n + \dots + \frac{t_1^{(1)}}{t_1^{(1)}} t + d_0, & f_2(t) &= \frac{t_2^{(n)}}{t_2^{(1)}} t^n + \dots + \frac{t_2^{(1)}}{t_2^{(1)}} t; \\ \bar{f}_1(t) &= \frac{\bar{t}_1^{(n)}}{\bar{t}_1^{(1)}} t^n + \dots + \frac{\bar{t}_1^{(1)}}{\bar{t}_1^{(1)}} t + d_0, & \bar{f}_2(t) &= \frac{\bar{t}_2^{(n)}}{\bar{t}_2^{(1)}} t^n + \dots + \frac{\bar{t}_2^{(1)}}{\bar{t}_2^{(1)}} t, \\ \tilde{p}(t) &= f_1(t)\bar{f}_1(t) + f_2(t)\bar{f}_2(t). \end{aligned}$$

Make sure the coefficients of $\tilde{p}(t)$ are real.

Step 2. Compute all distinct zeros (real or complex) of the discriminant polynomial $\tilde{p}(t)$ (in MATLAB, use the command `roots`). Denote these zeros by $\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_k, \bar{\eta}_1, \dots, \bar{\eta}_k$ such that ξ_1, \dots, ξ_s are distinct real numbers and η_1, \dots, η_k are distinct nonreal complex numbers. Then classify η_1, \dots, η_k into two sets T_1 and T_2 :

$$\begin{aligned} T_1 &= \{\eta \in \{\eta_1, \dots, \eta_k\} \mid f_1(\eta) = f_2(\eta) = \bar{f}_1(\eta) = \bar{f}_2(\eta) = 0\}, \\ T_2 &= \{\eta_1, \dots, \eta_k\} \setminus T_1. \end{aligned}$$

When the simple quaternionic polynomial considered is a polynomial with real coefficients, then directly set $T_1 = \{\eta_1, \dots, \eta_k\}$ and $T_2 = \emptyset$.

Step 3. For each $\eta_i \in T_2$, compute $f_1(\eta_i)$, $f_2(\eta_i)$. Then by Formula (2.10), output all roots of $q_n x^n + \dots + q_1 x + q_0 = 0$.

If we use our Theorem 4, then we get a better version of Algorithm 1.

Algorithm 1' (for solving the simple quaternionic polynomial equation $q_n x^n + \dots + q_1 x + q_0 = 0$)

Step 1. Write the equation as $p(x) := p_n x^n + \dots + p_1 x + d_0 = 0$ with $d_0 = 0$ or 1 (in fact, if $q_0 \neq 0$, simply multiply the equation by q_0^{-1} on the left). Then write $p(t) = g(t)(g_1(t) + g_2(t)\mathbf{j})$ where t is considered as a real variable, $g, g_1, g_2 \in \mathbb{C}[t]$ with $\gcd(g_1, g_2) = 1$. Now we compute $\tilde{g}(t) = g_1(t)\bar{g}_1(t) + g_2(t)\bar{g}_2(t)$.

Step 2. Compute all distinct zeros (real or complex) for $g(t)$ and $\tilde{g}(t)$ respectively (in MATLAB, use the command `roots`):

first2

 (3.1) $\{\xi_1, \dots, \xi_s; \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \dots, \lambda_t, \bar{\lambda}_t; \eta_1, \dots, \eta_{k_1}\},$

sec2

 (3.2) $\{\eta_{k_1+1}, \bar{\eta}_{k_1+1}, \dots, \eta_k, \bar{\eta}_k\},$

where ξ_1, \dots, ξ_s are distinct real numbers; $\{\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \dots, \lambda_t, \bar{\lambda}_t\}$ and $\{\eta_1, \dots, \eta_{k_1}, \eta_{k_1+1}, \bar{\eta}_{k_1+1}, \dots, \eta_k, \bar{\eta}_k\}$ are two sets of distinct nonreal complex numbers. Delete those η_i 's and $\bar{\eta}_i$'s in (3.2) if one of them appears in the set (3.1).

Step 3. For each η_i , we compute

formulaomega

 (3.3)
$$\omega_i = \frac{1}{|g_1(\bar{\eta}_i)|^2 + |g_2(\bar{\eta}_i)|^2} \{ |g_2(\bar{\eta}_i)|^2 \bar{\eta}_i + |g_1(\bar{\eta}_i)|^2 \eta_i + 2g_2(\bar{\eta}_i)\overline{g_1(\bar{\eta}_i)}(\text{Im}\eta_i)\mathbf{k} \}.$$

Then the quaternionic solution set of $p(x)$ is

formula21

 (3.4) $\{\xi_1, \dots, \xi_s, \omega_1, \dots, \omega_k\} \dot{\cup}_{i=1}^t [\lambda_i].$

Prior to our method, Janovská and Opfer gave an algorithm in [9] for solving the same quaternionic equation $q_n x^n + \dots + q_1 x + q_0 = 0$ ($q_n \neq 0$). In order to introduce their algorithm precisely, let us first recall the known concept, companion polynomial. For

$$p(x) = \sum_{j=0}^n q_j x^j, q_j \in \mathbb{H}, j = 0, \dots, n, q_0, q_n \neq 0,$$

following Niven [14], or more recently Janovská and Opfer [9], its *companion polynomial* is defined by

companion

(3.5)

$$q_{2n}(x) = \sum_{j,k=0}^n \bar{q}_j q_k x^{j+k} = \sum_{k=0}^{2n} b_k x^k, \text{ where } b_k = \sum_{j=\max(0,k-n)}^{\min(k,n)} \bar{q}_j q_{k-j} \in \mathbb{R}.$$

We remark that q_{2n} is equal to the discriminant $\tilde{p}(t)$ in our case. From Pogorui and Shapiro[16], we know that all powers x^j , $j \in \mathbb{Z}$ of a quaternion x have the form $x^j = \alpha x + \beta$ with real α, β . In particular, $x^2 = 2(\text{Re } x) x - |x|^2$. In order to determine the numbers α, β , Janovská and Opfer in [9] set up the following iteration:

eq: iteration1

$$(3.6) \quad \begin{cases} x^j = \alpha_j x + \beta_j, \alpha_j, \beta_j \in \mathbb{R}, j = 0, 1, \dots, \\ \alpha_0 = 0, \beta_0 = 1, \\ \alpha_{j+1} = 2\text{Re } x \alpha_j + \beta_j, \\ \beta_{j+1} = -|x|^2 \alpha_j, j = 0, 1, \dots \end{cases}$$

Now by means of the first line of Iteration (3.6), the polynomial $p(x)$ can be rewritten as

$$p(x) = \sum_{j=0}^n q_j (\alpha_j x + \beta_j) = \left(\sum_{j=0}^n q_j \alpha_j \right) x + \left(\sum_{j=0}^n q_j \beta_j \right) \equiv A(x)x + B(x),$$

where

formula3

(3.7)

$$A(x) = \sum_{j=0}^n q_j \alpha_j, B(x) = \sum_{j=0}^n q_j \beta_j.$$

With these in hand, we can state the algorithm given in [9], as follows.

Algorithm 2 (for solving the simple quaternionic equation $q_n x^n + \dots + q_1 x + q_0 = 0$)

Step 1. Write $q_n x^n + \dots + q_1 x + q_0 = 0$ as $p(x) = a_n x^n + \dots + a_1 x + a_0 = 0$ with $a_n = 1$. For this $p(x)$, compute the real coefficients b_0, b_1, \dots, b_{2n} of the companion polynomial $q_{2n}(x)$ by formula (3.5).

Step 2. Compute all $2n$ (real and complex) zeros of $q_{2n}(x)$, denote these zeros by z_1, z_2, \dots, z_{2n} and order them (if necessary) such that $z_{2j-1} = \overline{z_{2j}}$,

$j = 1, 2, \dots, n$.

Step 3. Define an integer vector **ind** (like *indicator*) of length n , and set all components to zero. Define a quaternionic vector Z of length n , and set all components to zero.

For $j := 1 : n$ **do**

(a) **Put** $z := z_{2j-1}$.

(b) **if** z is real, $Z(j) := z$; go to the next step; **end if**

(c) **Compute** $v := \overline{A(z)}B(z)$ by formula (3.7), with the help of Iteration (3.6).

(d) **if** $v = 0$, put **ind**(j) := 1; $Z(j) := z$; go to the next step; **end if**

(e) **if** $v \neq 0$, let $(v_1, v_2, v_3, v_4) := v$. Compute $|w| := \sqrt{v_2^2 + v_3^2 + v_4^2}$, and put

$$\boxed{\text{formula4}} \quad (3.8) \quad Z(j) := \left(\operatorname{Re} z, -\frac{|\operatorname{Im} z|}{|w|} v_2, -\frac{|\operatorname{Im} z|}{|w|} v_3, -\frac{|\operatorname{Im} z|}{|w|} v_4 \right).$$

end if

end for

In this algorithm, corresponding to a real z the expression $Z(j)$ produces a real isolated zero z , corresponding to “ $v = 0$ ” the expression $Z(j)$ produces a spherical zero $[z]$, and corresponding to “ $v \neq 0$ ” the expression $Z(j)$ produces an isolated zero $\operatorname{Re} z - \frac{|\operatorname{Im} z|}{|w|} v_2 \mathbf{i} - \frac{|\operatorname{Im} z|}{|w|} v_3 \mathbf{j} - \frac{|\operatorname{Im} z|}{|w|} v_4 \mathbf{k}$. The output results of $Z(j)$ produce all zeros of polynomial $q_n x^n + \dots + q_1 x + q_0$. Algorithm 2’s original edition is Section 7 of [9], where $|w| := \sqrt{v_2^2 + v_3^2 + v_4^2}$ is false, which is a misprint.

4. EXAMPLES

Example 1 In \mathbb{H} , solve the equation $p(x) := \mathbf{i}x^3 + \mathbf{j}x^2 + \mathbf{k}x + 1 = 0$.

Solution 1. Use our Algorithm 1 to do this. Write the coefficients $\mathbf{i}, \mathbf{j}, \mathbf{k}$ into the form: $\mathbf{i} = \mathbf{i} + 0\mathbf{j}$, $\mathbf{j} = 0 + 1\mathbf{j}$, $\mathbf{k} = 0 + \mathbf{i}\mathbf{j}$. Then the derived polynomials of this equation are

$$\begin{aligned} f_1(t) &= \mathbf{i}t^3 + 1, & f_2(t) &= t^2 + \mathbf{i}t, \\ \bar{f}_1(t) &= -\mathbf{i}t^3 + 1, & \bar{f}_2(t) &= t^2 - \mathbf{i}t, \end{aligned}$$

the discriminant polynomial is

$$\tilde{p}(t) = f_1 \bar{f}_1 + f_2 \bar{f}_2 = (t - \mathbf{i})(t + \mathbf{i})(t - e^{\mathbf{i}\frac{\pi}{4}})(t - e^{\mathbf{i}\frac{3\pi}{4}})(t - e^{\mathbf{i}\frac{5\pi}{4}})(t - e^{\mathbf{i}\frac{7\pi}{4}}),$$

which has no real root. It is easy to see that

$$T_1 = \emptyset \text{ and } T_2 = \{\eta_1 = \mathbf{i}, \eta_2 = e^{\mathbf{i}\frac{\pi}{4}}, \eta_3 = e^{\mathbf{i}\frac{3\pi}{4}}\}.$$

For $\eta_1 = \mathbf{i}$, then $f_1(\mathbf{i}) = 2 \neq 0$ and $f_2(\mathbf{i}) = -2$, and we get an isolated zero by Formula (2.10): $\frac{1}{8} \cdot (4\mathbf{i} - 4\mathbf{i} - 2 \cdot (-2) \cdot 2 \cdot 1 \cdot \mathbf{k}) = \mathbf{k}$;

For $\eta_2 = e^{\mathbf{i}\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathbf{i}$, then $f_1(\eta_2) = \frac{2-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathbf{i}$ and $f_2(\eta_2) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}+2}{2}\mathbf{i}$, and we get another isolated zero by Formula (2.10): $\frac{\sqrt{2}}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{k}$;

Similarly for $\eta_3 = e^{\mathbf{i}\frac{3\pi}{4}}$ we get the isolated zero: $\frac{\sqrt{2}}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{k}$.

Thus, the solution set is $\left\{ \mathbf{k}, \frac{\sqrt{2}}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{k}, -\frac{\sqrt{2}}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{k} \right\}$.

Solution 2. Use our Algorithm 2 to do this. Write $p(t) = (\mathbf{i}t^3 + 1) + (t^2 + \mathbf{i}t)\mathbf{j} := f_1 + f_2\mathbf{j}$, then

$$g = \gcd(f_1, f_2) = t + \mathbf{i}, g_1 = \mathbf{i}t^2 + t - \mathbf{i}, g_2 = t, \tilde{g} = g_1\bar{g}_1 + g_2\bar{g}_2 = t^4 + 1,$$

Compute the zeros of g : $\eta_1 = -\mathbf{i}$.

Compute the zeros of \tilde{g} : $\eta_2 = e^{\mathbf{i}\frac{\pi}{4}}, \bar{\eta}_2, \eta_3 = e^{\mathbf{i}\frac{3\pi}{4}}, \bar{\eta}_3$.

Now for $\eta_i (i = 1, 2, 3)$, compute $g_1(\bar{\eta}_i)$, $|g_1(\bar{\eta}_i)|^2$, $g_2(\bar{\eta}_i)$, and $|g_2(\bar{\eta}_i)|^2$, by formula (3.3) we get the solution set of $\mathbf{i}x^3 + \mathbf{j}x^2 + \mathbf{k}x + 1 = 0$:

$$\left\{ \mathbf{k}, \frac{\sqrt{2}}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{k}, -\frac{\sqrt{2}}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{k} \right\}.$$

Example 2 Solve the equation $x^3 + x^2 + x + 1 = 0$ in \mathbb{H} .

Solution 1. We will use our method to do this first. Since $x^3 + x^2 + x + 1 = (x + 1)(x + \mathbf{i})(x - \mathbf{i})$, from Corollary 3(a), we directly know the solution set is $\{-1\} \cup [\mathbf{i}]$.

Solution 2. Now we use the method in [9] to do this.

First Step. By formula (3.5), we get

$$q_6(x) = (x^3 + x^2 + x + 1)^2 = x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1.$$

Second Step. Compute all zeros of $q_6(x)$: $-1, -1, \mathbf{i}, \mathbf{i}, -\mathbf{i}, -\mathbf{i}$.

Third Step. For -1 we get a real isolated zero -1 . For \mathbf{i} we need give the following expansion by Iteration (3.6):

$$\mathbf{i}^0 = 0\mathbf{i} + 1, \mathbf{i}^1 = 1\mathbf{i} + 0, \mathbf{i}^2 = 0\mathbf{i} + (-1), \mathbf{i}^3 = (-1)\mathbf{i} + 0.$$

Then by formula (3.7) get

$$A(\mathbf{i}) = 0 + 1 + 0 + (-1) = 0, B(\mathbf{i}) = 1 + 0 + (-1) + 0 = 0,$$

and $v := \overline{A(\mathbf{i})}B(\mathbf{i}) = 0$. It leads to a spherical zero $[\mathbf{i}]$. Finally, for $-\mathbf{i}$ we have to repeat the same process as that made for \mathbf{i} :

$$(-\mathbf{i})^0 = 0(-\mathbf{i}) + 1, (-\mathbf{i})^1 = 1(-\mathbf{i}) + 0,$$

$$(-\mathbf{i})^2 = 0(-\mathbf{i}) + (-1), (-\mathbf{i})^3 = -1(-\mathbf{i}) + 0,$$

$$A(-\mathbf{i}) = 0 + 1 + 0 + (-1) = 0, B(-\mathbf{i}) = 1 + 0 + (-1) + 0 = 0,$$

and $v := \overline{A(-\mathbf{i})}B(-\mathbf{i}) = 0$. It also produces a spherical zero $[-\mathbf{i}]$. Note that $[-\mathbf{i}] = [\mathbf{i}]$, so the solution set is $\{-1\} \cup [\mathbf{i}]$.

At last let us solve the same polynomial in Example 3.8 of [9].

Example 3. Find all zeros of $p(z) = z^6 + \mathbf{j}z^5 + \mathbf{i}z^4 - z^2 - \mathbf{j}z - \mathbf{i}$.

Solution 1. We first use the method in this paper to do this.

We have $\mathbf{i}p(z) = \mathbf{i}z^6 + \mathbf{i}\mathbf{j}z^5 - z^4 - \mathbf{i}z^2 - \mathbf{i}\mathbf{j}z + 1$. Then

$$\mathbf{i}p(t) = (\mathbf{i}t^6 - t^4 - \mathbf{i}t^2 + 1) + (\mathbf{i}t^5 - \mathbf{i}t)\mathbf{j},$$

$$g = t^4 - 1, g_1 = \mathbf{i}t^2 - 1, g_2 = \mathbf{i}t,$$

$$\tilde{g} = g_1\bar{g}_1 + g_2\bar{g}_2 = (\mathbf{i}t^2 - 1)(-\mathbf{i}t^2 - 1) + \mathbf{i}t \cdot (-\mathbf{i}t) = t^4 + t^2 + 1.$$

The all distinct zeros (real or complex) for g and \tilde{g} are respectively $\{1, -1, \mathbf{i}, -\mathbf{i}\}$ and $\{e^{-\mathbf{i}\frac{\pi}{3}}, e^{\mathbf{i}\frac{\pi}{3}}, e^{-\mathbf{i}\frac{2\pi}{3}}, e^{\mathbf{i}\frac{2\pi}{3}}\}$. Now we take

$$\begin{aligned}\eta_1 &= e^{-\mathbf{i}\frac{\pi}{3}} = 1/2 - \sqrt{-3}/2 \\ \eta_2 &= e^{-\mathbf{i}\frac{2\pi}{3}} = -1/2 - \sqrt{-3}/2, \\ g_1(\bar{\eta}_1) &= \mathbf{i}e^{\mathbf{i}\frac{2\pi}{3}} - 1 = -1 - \frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{i}, |g_1(\bar{\eta}_1)|^2 = 2 + \sqrt{3}; \\ g_2(\bar{\eta}_1) &= \mathbf{i}e^{\mathbf{i}\frac{\pi}{3}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i}, |g_2(\bar{\eta}_1)|^2 = 1; \\ g_1(\bar{\eta}_2) &= \mathbf{i}e^{\mathbf{i}\frac{4\pi}{3}} - 1 = -1 + \frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{i}, |g_1(\bar{\eta}_2)|^2 = 2 - \sqrt{3}; \\ g_2(\bar{\eta}_2) &= \mathbf{i}e^{\mathbf{i}\frac{2\pi}{3}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{i}, |g_2(\bar{\eta}_2)|^2 = 1.\end{aligned}$$

So by formula (3.3), we get

$$\begin{aligned}\omega_1 &= \frac{1}{3 + \sqrt{3}} \left\{ \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i} \right) + (2 + \sqrt{3}) \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i} \right) \right. \\ &\quad \left. + 2 \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i} \right) \left(-1 - \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i} \right) \left(-\frac{\sqrt{3}}{2} \right) \mathbf{k} \right\} \\ &= \frac{1}{2} - \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k} \\ \omega_2 &= \frac{1}{3 - \sqrt{3}} \left\{ \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i} \right) + (2 - \sqrt{3}) \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i} \right) \right. \\ &\quad \left. + 2 \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{i} \right) \left(-1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i} \right) \left(-\frac{\sqrt{3}}{2} \right) \mathbf{k} \right\} \\ &= -\frac{1}{2} + \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}.\end{aligned}$$

Hence, the set of roots of $p(z)$ is

$$\{1, -1\} \cup \left\{ \frac{1}{2} - \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}, -\frac{1}{2} + \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k} \right\} \cup [\mathbf{i}].$$

Solution 2. We use the method in [9] to redo this.

Step 1 Compute the companion polynomial $q_{12}(x)$ by formula (3.5):

$$\begin{aligned}q_{12}(x) &= x^{12} + \mathbf{j}x^{11} + \mathbf{i}x^{10} - x^8 - \mathbf{j}x^7 - \mathbf{i}x^6 \\ &\quad - \mathbf{j}x^{11} + x^{10} - \mathbf{j}\mathbf{i}x^9 + \mathbf{j}x^7 - x^6 + \mathbf{j}\mathbf{i}x^5 \\ &\quad - \mathbf{i}x^{10} - \mathbf{k}x^9 + x^8 + \mathbf{i}x^6 + \mathbf{k}x^5 - x^4 \\ &\quad - x^8 - \mathbf{j}x^7 - \mathbf{i}x^6 + x^4 + \mathbf{j}x^3 + \mathbf{i}x^2 \\ &\quad + \mathbf{j}x^7 - x^6 + \mathbf{j}\mathbf{i}x^5 - \mathbf{j}x^3 + x^2 - \mathbf{j}\mathbf{i}x \\ &\quad + \mathbf{i}x^6 + \mathbf{k}x^5 - x^4 - \mathbf{i}x^2 - \mathbf{k}x + 1 \\ &= x^{12} + 0 + x^{10} + 0 - x^8 + 0 - 2x^6 + 0 - x^4 + 0 + x^2 + 1 \\ &= x^{12} + x^{10} - x^8 - 2x^6 - x^4 + x^2 + 1.\end{aligned}$$

Step 2 Compute the zeros of $q_{12}(x)$.

$$\begin{aligned}
 q_{12}(x) &= x^8(x^4 + x^2 - 1) - x^6 - x^4 + x^2 - x^6 + 1 \\
 &= x^8(x^4 + x^2 - 1) - x^2(x^4 + x^2 - 1) - x^6 + 1 \\
 &= (x^4 + x^2 - 1)(x^8 - x^2) - (x^6 - 1) \\
 &= (x^4 - 1)(x^2 + 1)(x^6 - 1).
 \end{aligned}$$

The 12 zeros of q_{12} are

$$1, 1, -1, -1, \mathbf{i}, \mathbf{i}, -\mathbf{i}, -\mathbf{i}, \frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}, \frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}, -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}, -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}.$$

Step 3 For 1 and -1 , we get isolated zeros: 1, -1 .

For $x = \mathbf{i}, -\mathbf{i}$, then

$$\begin{aligned}
 \mathbf{i}^0 &= 0\mathbf{i} + 1 & (-\mathbf{i})^0 &= 0(-\mathbf{i}) + 1 \\
 \mathbf{i}^1 &= 1\mathbf{i} + 0 & (-\mathbf{i})^1 &= 1(-\mathbf{i}) + 0 \\
 \mathbf{i}^2 &= 0\mathbf{i} + (-1) & (-\mathbf{i})^2 &= 0(-\mathbf{i}) + (-1) \\
 \mathbf{i}^3 &= (-1)\mathbf{i} + 0 & (-\mathbf{i})^3 &= (-1)(-\mathbf{i}) + 0 \\
 \mathbf{i}^4 &= 0\mathbf{i} + 1 & (-\mathbf{i})^4 &= 0(-\mathbf{i}) + 1 \\
 \mathbf{i}^5 &= 1\mathbf{i} + 0 & (-\mathbf{i})^5 &= 1(-\mathbf{i}) + 0 \\
 \mathbf{i}^6 &= 0\mathbf{i} + (-1) & (-\mathbf{i})^6 &= 0(-\mathbf{i}) + (-1)
 \end{aligned}$$

$$\begin{aligned}
 A(\mathbf{i}) &= 1 \cdot 0 + \mathbf{j} \cdot 1 + \mathbf{i} \cdot 0 & A(-\mathbf{i}) &= 1 \cdot 0 + \mathbf{j} \cdot 1 + \mathbf{i} \cdot 0 \\
 &+ 0 \cdot (-1) + (-1) \cdot 0 & &+ 0 \cdot (-1) + (-1) \cdot 0 \\
 &+ (-\mathbf{j}) \cdot 1 + (-\mathbf{i}) \cdot 0 = 0 & &+ (-\mathbf{j}) \cdot 1 + (-\mathbf{i}) \cdot 0 = 0
 \end{aligned}$$

$$v = \overline{A(\mathbf{i})}B(\mathbf{i}) = 0 \quad v = \overline{A(-\mathbf{i})}B(-\mathbf{i}) = 0.$$

They produce the same spherical zero $[\mathbf{i}]$ since $[-\mathbf{i}] = [\mathbf{i}]$.

For $x = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}, \frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}$, we have

$$\begin{aligned}
 (\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i})^0 &= 0(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}) + 1 & (\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i})^0 &= 0(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}) + 1 \\
 (\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i})^1 &= 1(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}) + 0 & (\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i})^1 &= 1(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}) + 0 \\
 (\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i})^2 &= 1(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}) + (-1) & (\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i})^2 &= 1(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}) + (-1) \\
 (\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i})^3 &= 0(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}) + (-1) & (\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i})^3 &= 0(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}) + (-1) \\
 (\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i})^4 &= (-1)(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}) + 0 & (\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i})^4 &= (-1)(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}) + 0 \\
 (\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i})^5 &= (-1)(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}) + 1 & (\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i})^5 &= (-1)(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}) + 1 \\
 (\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i})^6 &= 0(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}) + 1 & (\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i})^6 &= 0(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}) + 1
 \end{aligned}$$

$$\begin{aligned}
 A(x) &= 1 \cdot 0 + \mathbf{j} \cdot -1 + \mathbf{i} \cdot -1 & A(x) &= 1 \cdot 0 + \mathbf{j} \cdot -1 + \mathbf{i} \cdot -1 \\
 &+ 0 \cdot 0 - 1 \cdot 1 + -\mathbf{j} \cdot 1 + -\mathbf{i} \cdot 0 & &+ 0 \cdot 0 + \dots \\
 &= -1 - \mathbf{i} - 2\mathbf{j} & &= -1 - \mathbf{i} - 2\mathbf{j}
 \end{aligned}$$

$$\begin{aligned}
 B(x) &= 1 \cdot 1 + \mathbf{j} \cdot 1 + \mathbf{i} \cdot 0 & B(x) &= 1 \cdot 1 + \mathbf{j} \cdot 1 \\
 &- 1 \cdot -1 + -\mathbf{j} \cdot 0 + -\mathbf{i} \cdot 1 + 0 \cdot -1 & &+ \mathbf{i} \cdot 0 + \dots \\
 &+ 0 \cdot -1 = 2 - \mathbf{i} + \mathbf{j} & &= 2 - \mathbf{i} + \mathbf{j}
 \end{aligned}$$

$$\begin{aligned}
 v = \overline{A(x)}B(x) &= -3 + 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} & v = \overline{A(x)}B(x) &= -3 + 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \\
 |w| &= 3\sqrt{3} & |w| &= 3\sqrt{3}.
 \end{aligned}$$

They produce the same isolated zero: $\frac{1}{2} - \frac{\sqrt{3}}{3\sqrt{3}} \cdot 3\mathbf{i} - \frac{\sqrt{3}}{3\sqrt{3}} \cdot 3\mathbf{j} - \frac{\sqrt{3}}{3\sqrt{3}} \cdot 3\mathbf{k}$, which is $0.5 - 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k}$.

Finally for $x = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}$, we still have to finish the following work:

$$\begin{aligned} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right)^0 &= 0\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right) + 1 & \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right)^0 &= 0\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right) + 1 \\ \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right)^1 &= 1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right) + 0 & \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right)^1 &= 1\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right) + 0 \\ \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right)^2 &= -1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right) + (-1) & \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right)^2 &= -1\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right) + (-1) \\ \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right)^3 &= 0\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right) + 1 & \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right)^3 &= 0\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right) + 1 \\ \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right)^4 &= 1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right) + 0 & \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right)^4 &= 1\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right) + 0 \\ \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right)^5 &= (-1)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right) + (-1) & \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right)^5 &= (-1)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right) + (-1) \\ \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right)^6 &= 0\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right) + 1 & \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right)^6 &= 0\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}\right) + 1 \end{aligned}$$

$$\begin{aligned} A(x) &= 1 \cdot 0 + \mathbf{j} \cdot -1 + \mathbf{i} \cdot 1 + 0 \cdot 0 \\ &\quad -1 \cdot -1 + -\mathbf{j} \cdot 1 + -\mathbf{i} \cdot 0 \\ &= 1 + \mathbf{i} - 2\mathbf{j} \end{aligned}$$

$$\begin{aligned} A(x) &= 1 \cdot 0 + \mathbf{j} \cdot -1 + \mathbf{i} \cdot 1 \\ &\quad + 0 \cdot 0 + \cdots \\ &= 1 + \mathbf{i} - 2\mathbf{j} \end{aligned}$$

$$\begin{aligned} B(x) &= 1 \cdot 1 + \mathbf{j} \cdot -1 + \mathbf{i} \cdot 0 + 0 \cdot 1 \\ &\quad -1 \cdot -1 + -\mathbf{j} \cdot 0 + -\mathbf{i} \cdot 1 \\ &= 2 - \mathbf{i} - \mathbf{j} \end{aligned}$$

$$\begin{aligned} B(x) &= 1 \cdot 1 + \mathbf{j} \cdot -1 + \mathbf{i} \cdot 0 \\ &\quad + 0 \cdot 1 + \cdots \\ &= 2 - \mathbf{i} - \mathbf{j} \end{aligned}$$

$$v = \overline{A(x)}B(x) = 3 - 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$v = \overline{A(x)}B(x) = 3 - 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$|w| = 3\sqrt{3}$$

$$|w| = 3\sqrt{3}.$$

By formula (3.8), They produce the same isolated zero: $-0.5 + 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k}$. The set of roots of $p(z)$ is

$$\{1, -1, 0.5 - 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k}, -0.5 + 0.5\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{k}\} \cup [\mathbf{i}].$$

In Example 3.8 of [9], the authors said that the 12 zeros of q_{12} were 1 (twice), -1 (twice), $\pm\mathbf{i}$ (twice each), $0.5(\pm 1 \pm \mathbf{i})$. We point out that $0.5(\pm 1 \pm \mathbf{i})$ should be corrected as $0.5(\pm 1 \pm \sqrt{3}\mathbf{i})$.

5. NUMERICAL CONSIDERATION AND ALGORITHM COMPARISON

The polynomial in Example 1 has the property that its discriminant polynomial \tilde{p} can be factored by hands as $(t - \mathbf{i})(t + \mathbf{i})(t - e^{\mathbf{i}\frac{\pi}{4}})(t - e^{\mathbf{i}\frac{3\pi}{4}})(t - e^{\mathbf{i}\frac{5\pi}{4}})(t - e^{\mathbf{i}\frac{7\pi}{4}})$. In general, one can not always expect to get the zeros of the discriminant polynomial by hands, one has to rely on machine computations. In Algorithm 1, if we compute the zeros of the discriminant polynomial, $\tilde{p}(x)$, of Example 3 in Section 4 by MATLAB 7.12.0(R2011a), we find the 12 zeros

are as following:

root table	(5.1)	1	−1.0000000000000001	+0.000000002066542 <i>i</i>
		2	−1.0000000000000001	−0.000000002066542 <i>i</i>
		3	−0.5000000000000000	+0.866025403784440 <i>i</i>
		4	−0.5000000000000000	−0.866025403784440 <i>i</i>
		5	0.999999990102304	+0.000000000000000 <i>i</i>
		6	1.000000009897694	−0.000000000000000 <i>i</i>
		7	0.5000000000000000	+0.866025403784439 <i>i</i>
		8	0.5000000000000000	−0.866025403784439 <i>i</i>
		9	0.00000000016075	+1.000000008531051 <i>i</i>
		10	0.00000000016075	−1.000000008531051 <i>i</i>
		11	−0.00000000016074	+0.999999991468949 <i>i</i>
		12	−0.00000000016074	−0.999999991468949 <i>i</i>

which are nearly same as Table 1 in [9]. This is simply because our $\tilde{p}(t)$ is exactly the companion polynomial $q_{12}(t)$. Hence, the similar measures to [9] can be made to obtain machine precision for the zeros with multiplicity 2 (e.g., by Newton's method). It is interesting to note that the discriminant polynomial given in this paper is always equal to the companion polynomial after considering the variable as a real variable.

In Algorithm 1, if we have found by MATLAB the zeros of discriminant polynomial, we usually by comparing $|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2$ with $|f_1(\bar{\eta}_i)|^2 + |f_2(\bar{\eta}_i)|^2$ decide which one ω_i should take between (2.11) and (2.12). If $|f_1(\eta_i)|^2 + |f_2(\eta_i)|^2$ is greater, then ω_i takes (2.11), otherwise takes (2.12). However, our Algorithm 1' does not need to make such a decision.

In Step 2 of Algorithm 1, we need decide whether a zero ξ is real. In our experience (the same as [9]), a test of the form $|\text{Im}\xi| < 10^{-5}$ is appropriate. There is another delicate decision to make in our Algorithm 1. That is, in Step 2 one has to decide whether a zero η satisfies $f_1(\eta) = f_2(\eta) = \bar{f}_1(\eta) = \bar{f}_2(\eta) = 0$. For this, a test for $f(\eta) = 0$ can be carried out in the form $|f(\eta)| < 10^{-10}$ (This was also used to test $v = 0$ in [9]). Our Algorithm 1' avoids making this delicate decision.

Both Algorithm 1 and Algorithm 2 need to find all zeros of the companion polynomial (note that the discriminant polynomial is always equal to the companion polynomial). Unlike Algorithm 2, our algorithms no longer need the use of any iterations. Using our algorithms, one can easily produce all zeros of the simple quaternionic polynomial from the zeros of the companion polynomial. Our algorithm has less workload, as shown in Section 4. This has a great advantage for a simple quaternionic polynomial with high degree. In fact, when one use Iteration (3.6) to compute $A(z)$ and $B(z)$ for a nonreal complex root z with $|z| \neq 1$, if the degree of the simple quaternionic polynomial considered is high, the workload is huge, even interferes one's deciding whether v is zero.

In [Page 252, 9] D. Janovská and G. Opfer said “We made some hundred tests with polynomials p_n of degree $n \leq 50$ with random integer coefficients in the range $[-5, 5]$ and with real coefficients in the range $[0, 1]$. In all cases we found only (nonreal) isolated zeros z . The test cases showed $|p_n(z)| \approx 10^{-13}$. Real zeros and spherical zeros did not show up. If n is too large, say $n \approx 100$, then usually it is not any more possible to find all zeros of the

companion polynomial by standard means (say roots in MATLAB) because the coefficients of the companion polynomial will be too large.” But this can be easily avoid by using Algorithm 1’ in many cases. For example, to solve a real coefficient polynomial of degree of 50, we can use Corollary 3 to do this easily.

Example 4. Find all zeros of $p(z) = z^{1000} - 2$ in \mathbb{H} .

Solution. We know that the complex solution set of $p(t)$ is

$$\{\pm \sqrt[1000]{2}\} \cup_{k=1}^{499} \{\eta_k = \sqrt[1000]{2} e^{\frac{k\pi i}{500}}, \bar{\eta}_k\}.$$

Applying Corollary 3, we obtain the solution set for $p(z)$:

$$\{\pm 1\} \cup_{k=1}^{499} [\sqrt[1000]{2} e^{\frac{k\pi i}{500}}].$$

According to the above comments D. Janovská and G. Opfer made on their method, it is impossible to find all solutions of this example since the degree of $p(z)$ is much larger than 100 and all roots except for $\pm \sqrt[1000]{2}$ are spherical.

6. CONCLUSION

The method introduced in Section 2 can also be used to find all zeros of simple polynomials with quaternionic coefficients located on only right side of the power of the variable, since $x^n q_n + \cdots + x q_1 + q_0 = 0$ is equivalent to $\overline{q_n} \overline{x}^n + \cdots + \overline{q_1} \overline{x} + \overline{q_0} = 0$. In retrospect, Eilenberg and Niven in [3] for the first time proved that quaternionic $q_n x^n + \cdots + q_1 x + q_0 = 0$ has always a root, but the method they used is topological. For this result, here we have provided a new alternate proof, without using any topological tools. Based on the derived polynomial and discriminant polynomial introduced in this paper, we have given the zeros in an explicit form for a simple quaternionic polynomial. Comparing with the methods provided in [9] to seek roots of $q_n x^n + \cdots + q_1 x + q_0 = 0$, our method is different, and has great advantages. Using our method, we have recovered several known results, and deduced several very interesting consequences concerning seeking the zeros of a simple quaternionic polynomial which do not seem to be deduced easily from the results in [9].

We would like to conclude our paper by making the following remark. We hope the method in the proof of our main theorem can be useful to seeking a method to find all solutions of a quaternionic polynomial equation with coefficients on both sides of the powers of the variable.

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